Rotation

Linear (translational) vs. Rotational motion

![Diagram of linear and rotational motion]

Angle of rotation: \( \theta = \frac{l}{r} \) in Radians

1 revolution \( \rightarrow 360° \)
1 revolution \( \rightarrow 2\pi \) rad

Example: Conversion

\[
1° = (1°)\left(\frac{2\pi \text{ rad}}{360°}\right) = 0.017444 \text{ rad} \\
1 \text{ rad} = (1 \text{ rad})\left(\frac{360°}{2\pi \text{ rad}}\right) = 57.296°
\]

Angular velocity: \( \omega = \frac{v}{r} \) in rad/s

\[
\omega = \frac{\Delta \theta}{\Delta t} \\
\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}
\]

Angular acceleration: \( \alpha = \frac{\alpha}{r} \) in rad/s\(^2\)

\[
\alpha = \frac{\Delta \omega}{\Delta t}
\]

Period: \( T \) in [s] Time it takes for 1 revolution

Frequency: \( f \) in [Hz] How many revolutions per 1 second

\[
f = \frac{1}{T} \\
\omega = 2\pi f
\]

Example 1 Determine the angular velocity of the Earth as it turns around its axis.
Example 2. A wheel spins at 600 rpm. What is its angular velocity in rad/s?

Example 3. A carrousel is initially at rest. At t=0 it is given a constant angular acceleration \( \alpha = 0.060 \text{ rad/s}^2 \), which increases its angular velocity for 8.0 s. At t=8.0 s, determine the following quantities:
(a) \( \omega \)
(b) linear velocity of the child, \( v \), located 2.5 m from the center
(c) tangential acceleration \( a_t \)
(d) the total centripetal acceleration
(d) the total linear acceleration of the child

\[
\begin{align*}
\alpha &= \frac{\omega_2 - \omega_1}{\Delta t} \\
\omega_2 &= (0.060 \text{ rad/s}^2)(8.0 \text{ s}) + (0 \text{ rad/s}) = 0.48 \text{ rad/s} \\
v &= \omega r = 1.2 \text{ m/s} \\
a &= \alpha r = 0.15 \text{ m/s}^2 \\
a_t &= \frac{v^2}{r} = 0.58 \text{ m/s}^2 \\
a &= \sqrt{a_t^2 + a_r^2} = 0.60 \text{ m/s}^2
\end{align*}
\]

Constant Angular Acceleration

<table>
<thead>
<tr>
<th>Linear</th>
<th>Rotational</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = \text{const} )</td>
<td>( \alpha = \text{const} )</td>
</tr>
<tr>
<td>( v = v_0 + at )</td>
<td>( \omega = \omega_0 + \alpha t )</td>
</tr>
<tr>
<td>( x = x_0 + v_0 t + \frac{at^2}{2} )</td>
<td>( \theta = \theta_0 + \omega_0 t + \frac{a \theta}{2} )</td>
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<tr>
<td>( v^2 = v_0^2 + 2a \Delta x )</td>
<td>( \omega^2 = \omega_0^2 + 2\alpha \Delta \theta )</td>
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</table>

Example 4. A centrifuge rotor is accelerated from rest to 20,000 rpm in 30 s
(a) What is its average angular acceleration?
(b) Through how many revolutions has the centrifuge rotor turned during its acceleration period, assuming constant angular acceleration?

\( \alpha \) = \frac{\Delta \omega}{\Delta t} = \frac{\omega - 0 \text{ rad/s}}{30 \text{s}} 
\]

20,000 revolutions per minute
1 min \( \rightarrow \) 20,000 revolutions
1 s \( \rightarrow \) \( \frac{20,000}{60} \) revolutions

\[
\begin{align*}
f &= 20,000 \frac{\text{rev}}{\text{min}} = 20,000 \frac{\text{rev}}{1 \text{ min}} \left( \frac{\text{min}}{60 \text{ s}} \right) = 333.33 \frac{\text{rev}}{\text{s}} \\
\omega &= 2\pi f = 2100 \text{ rad/s}
\end{align*}
\]
\[
\alpha = \frac{\Delta \omega}{\Delta t} = \frac{\omega - 0 \text{ rad}}{30 \text{ s}} = \frac{2100 \text{ rad}}{30 \text{ s}} = 70 \text{ rad/s}^2
\]

(b) \[
\theta = \theta_0 + \omega_0 t + \frac{\alpha t^2}{2}
\]
\[
\theta = 0 \text{ rad} + \left(0 \text{ rad/s}\right)(30 \text{ s}) + \frac{(70 \text{ rad/s}^2)(30 \text{ s})^2}{2} = 3.15 \times 10^4 \text{ rad}
\]

Revolutions: \[
\frac{3.15 \times 10^4 \text{ rad}}{2\pi} = 5.0 \times 10^3 \text{ rev.}
\]

OR:
\[
\omega^2 = \omega_0^2 + 2\alpha \Delta \theta
\]
\[
\theta - \theta_0 = \frac{\omega^2 - \omega_0^2}{2\alpha}
\]
\[
\theta = 0 \text{ rad} + \frac{(2100 \text{ rad})^2 - (0,0)^2}{2(70 \text{ rad/s}^2)} = 3.15 \times 10^4 \text{ rad}
\]

- Torque \( \tau = rF_{\perp} \) in [N.m]
  (ability of force to rotate objects)

  \[\tau = rF_{\perp} \text{ is the same as } \tau = rF\sin\theta\]

Newton's Second Law
\[\tau = I\alpha \quad (F = ma)\]

- \( r \) – lever arm
- \( I \) – moment of Inertia

Example 5. Determine the net torque on the wheel.

\[
\tau = r_a F_a - r_b F_b \sin(60^\circ) = -6.7 \text{ N.m}
\]

Moment of Inertia \( I = mr^2 \) in [kg.m^2]
(rotational inertia)
Table with the Rotational Inertia for Various Objects

Rolling motion without slipping

\[ \omega = \frac{v}{r} \]
\[ v_c = v \]

Rotational Kinetic Energy  \[ KE_{\text{rot}} = \frac{I\omega^2}{2} \]

Example 6. Sphere rolling down an incline
What will be the speed of a solid sphere of mass M and radius R when it reaches the bottom of an incline if it starts from rest at a vertical height H and rolls without slipping? (no slipping). Compare with object sliding down a frictionless incline.

<table>
<thead>
<tr>
<th></th>
<th>top</th>
<th>bottom</th>
</tr>
</thead>
<tbody>
<tr>
<td>KE (translation)</td>
<td>0</td>
<td>( \frac{1}{2}mv^2 )</td>
</tr>
<tr>
<td>KE (rotation)</td>
<td>0</td>
<td>( \frac{1}{2}I\omega^2 )</td>
</tr>
<tr>
<td>PE (gravitational)</td>
<td>( mgH )</td>
<td>0</td>
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</table>

\[ MgH + 0 + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \]

No slipping: \( v = \omega r \);  \( I_{\text{solid sphere}} = \frac{2}{5}MR^2 \)
\[ M g H = \frac{M v^2}{2} + \frac{2 M R^2 (\frac{\dot{\omega}}{R})}{2} = \frac{M v^2 + \frac{2}{5} M v^2}{2} = \frac{7}{10} M v^2 \quad \rightarrow v = \sqrt{\frac{10 g H}{7}} \]

compare with \( v = \sqrt{2 g H} \) for a sliding mass

**Work done by torque:**
\[ W = F \Delta l = F r \Delta \theta = \tau \Delta \theta \]

**Power:**
\[ P = \frac{\Delta E}{\Delta t} = \frac{W}{\Delta t} = \tau \frac{\Delta \theta}{\Delta t} = \tau \omega \]

------------------------------------------------------------------------------------------------------------------------

**Angular Momentum:** \( L = I \omega \) in [kg.m²/s]

\[ L = I \omega = m r^2 \frac{\dot{v}}{r} = m v r \]
\[ L = r \cdot p_\perp = r p \sin \theta \]

**Newton's Second Law:**
\[ \sum \tau = \frac{\Delta L}{\Delta t} = \frac{I \omega - I \omega_0}{\Delta t} = \frac{I \dot{\omega}}{\Delta t} = I \alpha \]

**Conservation of Angular Momentum:**
\[ \sum \tau = 0 \Rightarrow L = \text{const.} \]

**Example 7. Collapsing Star**
A star collapses. A star with a radius of \( 7 \times 10^5 \) km and mass \( M = 2.0 \, M_{\text{Sun}} \), makes one revolution every 10 days. If it collapses to a radius of 10 km, determine the new angular velocity with which it rotates.

Before the Collapse:
\[ R = 7 \times 10^5 \text{km} \]
\[ M = 2.0 \, M_{\text{Sun}} \]
\[ \text{speed} = 1.0 \text{revolutions per 10 days} \]
\[ \omega = \frac{1.0 \times 2\pi}{10 (24) (3600 \text{s})} = 7.27 \times 10^{-6} \text{rad/s} \]

After the collapse:
\[ R = 10 \text{km} \]
\[ \text{speed} = ? \]
\[ L = \text{const} \]
\[ I_b \omega_b = I_a \omega_a \]
\[ \omega_a = \frac{I_b}{I_a} \omega_b = \frac{\frac{3}{2}M(7 \times 10^9 \text{km})^2}{\frac{3}{2}M(10 \text{km})^2} \left( 7.27 \times 10^{-6} \text{rad/s} \right) = 35623 \frac{\text{rad}}{\text{s}} = 5.7 \times 10^3 \text{rev/s} \]
\[ \omega_a = \frac{3}{2}\frac{M(7 \times 10^9 \text{km})^2}{\text{2}} \left( \frac{1}{10} \text{ revolutions/day} \right) = 4.9 \times 10^8 \text{ revolutions/day} \]

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<tr>
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<th>Rotational</th>
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<td>( \text{kg} )</td>
<td>( I )</td>
</tr>
<tr>
<td>( F )</td>
<td>( \text{N} )</td>
<td>( \tau )</td>
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<td>( \text{W} )</td>
<td>( P = \frac{\Delta E}{\Delta t} = \tau \cdot \omega )</td>
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<tr>
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<td>( \text{kg.m} )</td>
<td>( L = I \omega )</td>
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<tr>
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<td>( \sum \tau = \frac{\Delta L}{\Delta t} )</td>
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